CONVERGENCE OF OPTIMAL CONTROL ALGORITHMS

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The algorithm that we will discuss solves optimal control algorithms of the form

$$(1) \quad \min g(u) \equiv \int_0^1 L(x(t,u),u(t),t)dt + h(x(1,u)), \text{ subject to the constraints}$$

$$(2) \quad \frac{d}{dt}x(t,u) = f(x(t,u), u(t), t), \ t \text{ in } [0,1], \text{ a.e.}$$

$$(3) \quad x(0,u) = x_0 \quad \text{where} \quad f{:}R^n \times R^m \times [0,1] \to R^n,$$

$L{:}R^n \times R^m \times [0,1] \to R^1$, $h{:}R^n \to R^1$, and u is a bounded measurable function. Let T be the interval [0,1].

ASSUMPTION 1:

The functions f and L above and their partial derivatives

$\frac{\partial}{\partial x}f, \frac{\partial}{\partial x}L$ exist and are continuous on $R^n \times R^m \times T$. The function h and its partial derivative $\frac{\partial}{\partial x}h$ exist and are continuous.

ASSUMPTION 2:

For each compact $\Omega \subseteq R^m$, there exists M > 0 such that

$\|f(x, u, t)\| \le M(\|x\| + 1)$ for all $(x,u,t) \ \varepsilon \ R^m \times \Omega \times T$ and

$\|f(x, u, t) - f(x', u, t)\| \le M\|x - x'\|$ for all $x, x' \ \varepsilon \ R^n$, $u \ \varepsilon \ \Omega$, $t \ \varepsilon \ T$.

DEFINITION 1:

Let V be the set of all non-negative unit measures (probability measures) on R^m. A ***relaxed control*** v is a function $v(\bullet): T \to V$ such that there exists some compact set U such that $\int_{R^m} dv(t) = 1 = \int_U dv(t)$ for all t in T.

DEFINITION 2:

Let v be a measure in V and let $\phi(\bullet, \bullet, \bullet)$ be a continuous function in (x, u, t). The symbol $\phi_r(x,u,t)$ denotes for fixed (x,t), the integral on R^m of $\phi(x,u,t)$ with respect to the probability measure v, i.e.

$$\phi_r(x,v,t) \equiv \int_{R^m} \phi(x,u,t)dv.$$

The relaxed problem is obtained from the original problem (1)-(3) by substituting the cost

(4) $$g_0(v) \equiv \int_0^1 L_r(x(t,v), v(t), t)dt + h_0(x(1,v))$$

for the cost (1) and the differential equation

(5) $$\dot{x}(t) = f_r(x(t), v(t), t) \equiv \int_{R_m} f(x(t), u, t)dv(t)$$

for the differential equation (2).

THEOREM 1:

Suppose that assumptions (1) and (2) are satisfied. Then for any

measurable relaxed control $v(\cdot)$ satisfying

$$\int_{R^m} dv(t) = 1 = \int_U dv(t) \text{ for all } t \text{ in } T, \text{ for some compact } U \subset R^m,$$

there exists an absolutely continuous function $x(\cdot, v)\colon T \to R^n$ that is the unique solution to (5), satisfying $x(0,v) = x_0$.

DEFINTION 3: Let $\lambda(\cdot,v):T \to R^n$ denote the solution of

(6) $$-\dot{\lambda}(t,v) = \left(\frac{\partial}{\partial x}H\right)_r^T (x(t,v), v(t), \lambda(t,v), t)$$

(7) $$\lambda(1,v) = \left(\frac{\partial}{\partial x}h\right)^T (x(1,v))$$

where $H:R^n \times R^m \times R^n \times T \to R^1$ is defined by

(8) $$H(x,u,\lambda,t) \equiv \lambda^T f(x,u,t) + L(x,u,t) \quad .$$

DEFINITION 4:

A sequence $\{v^i(\cdot)\}_0^\infty$ of relaxed controls converges in the sense of control measures (i.s.c.m) to a relaxed control $\bar{v}$ if for every continuous realvalued function g(t,u) defined on $T \times R^m$ and every subinterval Δ of T, the values $\int_\Delta g_r(t,v^i(t))dt$ converge to $\int_\Delta g_r(t,\bar{v}(t))dt$.

THEOREM 2:

Let $\{v^i(\cdot)\}_0^\infty$ be a sequence of relaxed controls such that for some compact set $U \subset R^m$, $\int_{R^m} dv^i(\cdot) \equiv 1 \equiv \int_U dv^i(\cdot)$ i = 0,1, ... Then there exists a relaxed control $\bar{v}(\cdot)$ and a subsequence indexed by a set $K \subset \{0,1,2, ...\}$ such that $\{v^i(\cdot)\}_{i \varepsilon K} \to \bar{v}(\cdot)$ i.s.c.m. .

DEFINITION 5:

If $\{(v^i, x^i, \lambda^i)\}_{i=0}^\infty$ is a sequence of relaxed controls, corresponding trajectories and multipliers such that $v^i \to \bar{v}$ i.s.c.m., $x^i \to \bar{x}$ uniformly, and $\lambda^i \to \bar{\lambda}$ uniformly, then we denote this by $\{(v^i, x^i, \lambda^i)\}_{i=0}^\infty \to (\bar{v}, \bar{x}, \bar{\lambda})$.

In the following lemmas, let U be an arbitrary compact set in R^m and let S be the set of measurable relaxed controls which vanish outside of U.

Lemma 1:

Let g be a continuous function from $R^p \times R^m \times T$ into R^q. Let $Y^i(\bullet)$, $\bar{Y}(\bullet)$ be continuous functions from T into R^m such that $Y^i(\bullet)$ converges to $\bar{Y}(\bullet)$ uniformly.Let $\{v^i(\cdot)\}_0^\infty$ be a sequence of relaxed controls in S that converges i.s.c.m to a relaxed control $\bar{v}(\bullet)$ in S. Then for each subinter val Δ of T,

(9) $$\int_\Delta g_r(Y^i(\tau), v^i(\tau), \tau)d\tau \to \int_\Delta g_r(\bar{Y}(\tau), \bar{v}(\tau), \tau)d\tau .$$

Lemma 2:

Let $\{y^i(\bullet)\}_{i\varepsilon I}$, where I is some indexing set , be a collection of absolutely continuous functions from T into R^n such that $\{y^i(0)\}_{i\varepsilon I}$ or $\{y^i(1)\}_{i\varepsilon I}$ is contained in a compact set of R^n. Let functions $Y^i\!:T \to R$, i in I, be defined by

(10) $$Y^i(t) = \|y^i(t)\|^2 + 1 .$$

If there exists an $M > 0$ such that $|\dot{Y}(t)| \leq M \cdot \dot{Y}(t)$ for almost all t in T, i in I

then the set $\{y^i(\bullet)\}_{i \varepsilon I}$ is equibounded and equicontinuous. If $I = \{0,1,...\}$, there exists a subsequence indexed by a set $K \subset \{0,1,2...\}$ and an absolutely continuous function $\bar{y}(\bullet)$ such that $y^i(\bullet)$ converges uniformly to $\bar{y}(\bullet)$ for i in K.

<u>Lemma 3</u>: If $\{x^i\}_{i=0}^{\infty}$ is a sequence of trajectories corresponding to a sequence of relaxed controls $\{v^i\}_{i=0}^{\infty} \subset S$, then $\{x^i\}_{i=0}^{\infty}$ is equibounded and equicontinuous, and there exists a subsequence indexed by a set $K \subset \{0,1,2...\}$, and an absolutely continuous $\bar{x}$ such that $\{x^i\}_{i \varepsilon K} \to \bar{x}$ uniformly.

<u>Lemma 4</u>: If $\{(v^i, x^i)\}_{i=0}^{\infty}$ is a sequence of relaxed controls and corresponding trajectories such that $\{v^i\}_{i=0}^{\infty} \subset S$ and $\{x^i\}_{i=0}^{\infty} \to \bar{x}$ uniformly, then the sequence of multipliers $\{\lambda^i\}_{i=0}^{\infty}$ is equibounded and equicontinuous, and there exists a subsequence of $\{(v^i, x^i)\}_{i=0}^{\infty}$, indexed by some set $K \subset \{0,1,2...\}$ and an absolutely continuous $\bar{\lambda}$ such that $\{\lambda^i\}_{i \varepsilon K} \to \bar{\lambda}$ uniformly.

Theorem 3: Let U be an arbitrary compact set in R^m and let S be the set of measurable relaxed controls which vanish outside of U. If

$\{(v^i, x^i, \lambda^i)\}_{i=0}^{\infty}$ is a sequence of relaxed controls, corresponding trajectories, and corresponding multipliers such that $\{v^i\}_{i=0}^{\infty} \subset S$, $\{v^i\} \to \bar{v}$ i.s.c.m., $\{x^i\} \to \bar{x}$ uniformly, and $\{\lambda^i\} \to \bar{\lambda}$ uniformly. Then $\bar{x} = x(\cdot,\bar{v})$ and $\bar{\lambda} = \lambda(\cdot,\bar{v})$. Furthermore, given a sequence $\{(v^i,x^i,\lambda^i)\}_{i=0}^{\infty}$ such that $\{v^i\} \subset S$, there always exist a subsequence that satisfies the above hypotheses and conclusions.

ARMIJO GRADIENT ALGORITHM

Step 0: Select a bounded measurable function u^0. Select a $\beta \varepsilon (0,1)$.

Step 1: Set i = 0.

Step 2: Compute x^i with $u = u^i$.

Step 3: Compute λ^i with $x = x^i$ and $u = u^i$.

Step 4: Compute $\nabla g(u^i)(\cdot) = \left(\frac{\partial}{\partial u}H\right)^T (x^i(\cdot),u^i(\cdot),\lambda^i(\cdot),\cdot)$.

Step 5: Set $\gamma = 1$.

Step 6: Compute

$$\theta(u^i,\gamma) \equiv \int_0^1 L(x(t, u^i - \gamma\nabla g(u^i)),u^i(t) - \gamma\nabla g(u^i)(t),t)dt \tag{11}$$

$$+ h(x(1,u^i - \gamma\nabla g(u^i))) - \int_0^1 L(x(t,u^i),u^i(t),t)dt - h(x(1,u^i))$$

$$+ \frac{\gamma}{2}\|\nabla g(u^i)\|_2^2 .$$

Step 7: If $\theta(u^i, \gamma) > 0$, set $\gamma = \beta\gamma$ and go to Step 6, else let

$$u^{i+1}(\cdot) = u^i(\cdot) - \gamma\nabla g(u^i(\cdot)) \quad \text{and go to Step 2.}$$

Let U be an arbitrary compact set $\subset R^m$, and let S be the set of measurable relaxed controls which vanish outside of U.

Definition 6: For any v in S, $\alpha \in R^1$, and $k \in C_m[T \times U]$, let $x(\cdot,v,\alpha,k)$ denote the solution of

$$(12) \quad \frac{d}{dt}x(t,v,\alpha,k) = \int_U f(x(t,v,\alpha,k),u + \alpha k(t,u),t)dv(t) \quad \text{a.e.}$$

$$(13) \quad x(0,v,\alpha,k) = x_0 .$$

Let $g(v,\alpha,k)$ denote

(14)

$$g(v,\alpha,k) = \int_0^1\left(\int_U L(x(t,v,\alpha,k),u + \alpha k(t,u),t)dv(t)\right)dt + h(x(1,v,\alpha,k)) .$$

Definition 7: Define $\nabla g(v)(t,u)$ and $\|\nabla g(v)\|_v$ by

$$(15) \quad \nabla g(v)(t,u) = \frac{\partial}{\partial u}H^T(x(t,v),u,\lambda(t,v),t)$$

(16) $$\|\nabla g(v)\|_v = \left[\int_0^1 \left(\int_U \|\nabla g(v)(t,u)\|^2 dv(t) \right) dt \right]^{1/2}$$

Lemma 5: Let $\{(v^i,x^i,\lambda^i)\}_{i=0}^{\infty}$ be a sequence of relaxed controls and corresponding trajectories and multipliers converging to $(\bar{v},\bar{x},\bar{\lambda})$ where $\{v^i\} \subset S$, $\bar{v} \in S$. Then for each $\alpha \in [-1,1]$, there exists an infinite subset $J(\alpha) \subset \{0,1,2,...\}$ such that $x(\cdot,v^i,\alpha,\nabla g(v^i)) \xrightarrow{J(\alpha)} x(\cdot,\bar{v},\alpha,\nabla g(\bar{v}))$.

Furthermore, $g(v^i,\alpha,\nabla g(v^i)) \xrightarrow{J(\alpha)} g(\bar{v},\alpha,\nabla g(\bar{v}))$.

Assumption 3: We will assume also that the functions f, L and h and their partial derivatives with respect to x and u up to second order exist and are continuous in (x, u, t).

Lemma 6: Let $\bar{v} \in S$ and $\bar{x},\bar{\lambda}$ be its corresponding trajectory and multiplier. Then there exists a P > 0 such that

(17) $$\left| g(\bar{v},\alpha,\nabla g(\bar{v})) - g(\bar{v},0,0) - \alpha\|\nabla g(\bar{v})\|_{\bar{v}}^2 \right| \le P\alpha^2 \quad \forall \alpha \in [-1,1].$$

In particular, there exists an integer $l(\bar{v})$ such that

(18) $$g(\bar{v},-\beta^{l(\bar{v})},\nabla g(\bar{v})) - g(\bar{v},0,0) + \frac{\beta^{l(\bar{v})}}{2}\|\nabla g(\bar{v})\|_{\bar{v}}^2 \le -\frac{\beta^{l(\bar{v})}}{4}\|\nabla g(\bar{v})\|_{\bar{v}}^2.$$

Lemma 7: Let $\{(v^i,x^i,\lambda^i)\}_{i=0}^{\infty}$ be a sequence of ordinary controls, corresponding trajectories and multipliers converging to $(\bar{v},\bar{x},\bar{\lambda})$ where $(v^i) \subset S$, $\bar{v} \in S$.

Suppose $\|\nabla g(\bar{v})\|_{\bar{v}}^2 > 0$. Then there exists a $\delta(\bar{v}) < 0$. an integer M > 0, and an infinite subset $K \subset (0,1,2,...)$ such that

(19) $g(v^i,-\gamma_i,\nabla g(v^i)) - g(v^i,0,0) \le \delta(\bar{v}) \quad \forall i \in K, \forall i \ge M$ where γ_i is the step size that the algorithm would construct in step 7 if $v = v^i$.

Theorem 4: Let $\{(v^i,x^i,\lambda^i)\}_{i=0}^{\infty}$ be a sequence of relaxed controls, corresponding trajectories and multipliers constructed by the algorithm. Suppose that there exists a compact set $U \subset R^m$ such that $\int_{R^m} dv^i(\cdot) \equiv 1 \equiv \int_U dv^i(\cdot)$ for all i. Let S be the set of relaxed controls that vanish outside of U. Then either the sequence is finite, in which case the last element is desireable, or it is infinite and every accumulation point of this sequence is desireable. i.e. $\|\nabla g(\bar{v})\|_{\bar{v}}^2 = 0$ if $(\bar{v},\bar{x},\bar{\lambda})$ is an accumulation point. Furthermore, at least one accumulation point exists.